Third and fourth degree collisional moments for inelastic Maxwell models

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Abstract

The third and fourth degree collisional moments for d-dimensional inelastic Maxwell models are exactly evaluated in terms of the velocity moments, with explicit expressions for the associated eigenvalues and cross coefficients as functions of the coefficient of normal restitution. The results are applied to the analysis of the time evolution of the moments (scaled with the thermal speed) in the free cooling problem. It is observed that the characteristic relaxation time toward the homogeneous cooling state decreases as the anisotropy of the corresponding moment increases. In particular, in contrast to what happens in the one-dimensional case, all the anisotropic moments of degree equal to or less than 4 vanish in the homogeneous cooling state for $d \geqslant 2$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A realistic model capturing the influence of dissipation on the dynamic properties of granular systems consists of a gas of inelastic hard spheres (IHS) with a constant coefficient of normal restitution $\alpha \leqslant 1$ [1]. For sufficiently low densities, the Boltzmann equation for IHS provides the adequate framework to describe the time evolution of the one-particle velocity distribution function $f(\mathbf{r}, \mathbf{v}; t)$ [2]. However, the intricacy of the Boltzmann collision operator for IHS makes it difficult to obtain exact results. For instance, the fourth cumulant a_2 of the velocity distribution in the so-called homogeneous cooling state (HCS) is not exactly known, although good estimates of it have been proposed [3–5]. For inhomogeneous situations, explicit expressions for the Navier–Stokes (NS) transport coefficients are approximately obtained by considering the leading terms in a Sonine polynomial expansion [6–9].

    As in the elastic case, part of the above difficulties can be overcome by considering the so-called Maxwell models, i.e., models for which the collision rate is independent of the relative

velocity of the two colliding particles. Inelastic Maxwell models (IMM) have attracted the attention of physicists and mathematicians since the beginning of the century [10–34]. The structure of the Boltzmann collision operator for IMM has the advantage of allowing for the derivation of a number of exact properties, such as the high-velocity tails [14–19] and the cumulants [15, 17, 22, 23, 27, 31] in homogeneous situations, the NS transport coefficients [27, 31] and the rheology under simple shear flow [13, 29]. As a consequence, it is possible to explore the influence of inelasticity on the dynamic properties in a clean way, without the need of introducing additional, and sometimes uncontrolled, approximations. Apart from their academic interest, it turns out that the IMM reliably describes the properties of IHS in some situations, as happens in the simple shear flow problem [29] and for the NS transport coefficients associated with the mass flux [31]. Furthermore, it is interesting to remark that recent experiments [35] for magnetic grains with dipolar interactions are well described by IMM.

The aim of this paper is to contribute to the advancement in the knowledge of exact properties of IMM by evaluating all the third and fourth degree moments of the Boltzmann collision operator for an arbitrary number of dimensions d. The knowledge of those collisional moments, along with that of the second degree collisional moments [27, 29], opens up a number of interesting applications. For instance, one can investigate the temporal relaxation toward the HCS, starting from arbitrary initial conditions (not necessarily isotropic), as measured by the lowest degree moments (namely, the fourth degree moments) which signal the non-Gaussian character of the asymptotic velocity distribution function. This issue will be covered in this paper.

This paper is organized as follows. In section 2 the Boltzmann equation for IMM is presented. Next, the Ikenberry polynomials [36] $Y_{2r|i_1i_2...i_s}(\mathbf{V})$ of degree $k = 2r + s$ are introduced and their associated collisional moments $J_{2r|i_1i_2...i_s}$ for $k = 3$ and 4 are evaluated, the technicalities being relegated to an appendix. The results are applied to the relaxation problem of the (scaled) moments toward their asymptotic values in the HCS in section 3. The paper is closed in section 4 with a brief discussion of the results obtained here.

2. Collisional moments for IMM

In the absence of external forces, the inelastic Boltzmann equation for a granular gas reads [1]

$$(\partial_t + \mathbf{v} \cdot \nabla) f(\mathbf{r}, \mathbf{v}; t) = J[\mathbf{v}|f, f], \tag{2.1}$$

where $J[\mathbf{v}|f, f]$ is the Boltzmann collision operator. The form of the operator J for IMM can be obtained from the form for IHS by replacing the IHS collision rate (which is proportional to the relative velocity of the two colliding particles) by an effective velocity-independent collision rate. With this simplification, the form of J becomes [28]

$$J[\mathbf{v}_1|f, f] = \frac{\nu}{n\Omega_d} \int d\mathbf{v}_2 \int d\widehat{\boldsymbol{\sigma}}[\alpha^{-1} f(\mathbf{v}_1') f(\mathbf{v}_2') - f(\mathbf{v}_1) f(\mathbf{v}_2)]. \tag{2.2}$$

Here,

$$n = \int d\mathbf{v} f(\mathbf{v}) \tag{2.3}$$

is the number density, ν is an effective collision frequency (to be chosen later), $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle in d dimensions, and $\alpha \leqslant 1$ refers to the constant coefficient of restitution. In addition, the primes on the velocities denote the initial values $\{\mathbf{v}_1', \mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1, \mathbf{v}_2\}$ following a binary collision:

$$\mathbf{v}_1' = \mathbf{v}_1 - \tfrac{1}{2}(1 + \alpha^{-1})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}, \qquad \mathbf{v}_2' = \mathbf{v}_2 + \tfrac{1}{2}(1 + \alpha^{-1})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}, \tag{2.4}$$

where $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity of the colliding pair and $\widehat{\boldsymbol{\sigma}}$ is a unit vector directed along the centers of the two colliding spheres. The collision frequency ν can be seen as a free parameter in the model. Its dependence on the coefficient of restitution α can be chosen to optimize the agreement with the results obtained from the Boltzmann equation for IHS. In particular, to get the same expression for the cooling rate as that found for IHS (evaluated in the local equilibrium approximation) one takes the choice [27]

$$\nu = \frac{d+2}{2}\nu_0, \qquad \nu_0 = \frac{4\Omega_d}{\sqrt{\pi}(d+2)}n\sigma^{d-1}\sqrt{\frac{T}{m}}, \tag{2.5}$$

where σ is the diameter of the spheres. Note that, in any case, the results derived in this paper will be independent of the specific choice of ν_0.

A useful identity for an arbitrary function $h(\mathbf{v})$ is given by

$$\mathcal{J}[h] \equiv \int d\mathbf{v}_1\, h(\mathbf{v}_1)J[\mathbf{v}_1|f,f] = \frac{\nu}{n\Omega_d}\int d\mathbf{v}_1\int d\mathbf{v}_2\, f(\mathbf{v}_1)f(\mathbf{v}_2)\int d\widehat{\boldsymbol{\sigma}}[h(\mathbf{v}_1'') - h(\mathbf{v}_1)], \tag{2.6}$$

where

$$\mathbf{v}_1'' = \mathbf{v}_1 - \tfrac{1}{2}(1+\alpha)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})\widehat{\boldsymbol{\sigma}} \tag{2.7}$$

denotes the post-collisional velocity. If $h(\mathbf{v})$ is a polynomial, then

$$\mathcal{M}[h] \equiv \int d\mathbf{v}\, h(\mathbf{v})f(\mathbf{v}) \tag{2.8}$$

is its associated velocity moment and $\mathcal{J}[h]$ is the corresponding collisional moment.

In the case of Maxwell models (both elastic and inelastic), it is convenient to introduce the Ikenberry polynomials [36] $Y_{2r|i_1i_2\ldots i_s}(\mathbf{V})$ of degree $k = 2r + s$, where $\mathbf{V} = \mathbf{v} - \mathbf{u}(\mathbf{r})$ is the peculiar velocity, $\mathbf{u}(\mathbf{r})$ being the mean flow velocity defined as

$$\mathbf{u} = \frac{1}{n}\int d\mathbf{v}\,\mathbf{v}f(\mathbf{v}). \tag{2.9}$$

The Ikenberry polynomials are defined as $Y_{2r|i_1i_2\ldots i_s}(\mathbf{V}) = V^{2r}Y_{i_1i_2\ldots i_s}(\mathbf{V})$, where $Y_{i_1i_2\ldots i_s}(\mathbf{V})$ is obtained by subtracting from $V_{i_1}V_{i_2}\ldots V_{i_s}$ that homogeneous symmetric polynomial of degree s in the components of $\mathbf{V}$ such as to annul the result of contracting the components of $Y_{i_1i_2\ldots i_s}(\mathbf{V})$ on any pair of indices. The polynomial functions $Y_{2r|i_1i_2\ldots i_s}(\mathbf{V})$ of degree smaller than or equal to 4 are

$$Y_{0|0}(\mathbf{V}) = 1, \qquad Y_{0|i}(\mathbf{V}) = V_i, \tag{2.10}$$

$$Y_{2|0}(\mathbf{V}) = V^2, \qquad Y_{0|ij}(\mathbf{V}) = V_iV_j - \frac{1}{d}V^2\delta_{ij}, \tag{2.11}$$

$$Y_{2|i}(\mathbf{V}) = V^2V_i, \qquad Y_{0|ijk}(\mathbf{V}) = V_iV_jV_k - \frac{1}{d+2}V^2(V_i\delta_{jk} + V_j\delta_{ik} + V_k\delta_{ij}), \tag{2.12}$$

$$Y_{4|0}(\mathbf{V}) = V^4, \qquad Y_{2|ij}(\mathbf{V}) = V^2\left(V_iV_j - \frac{1}{d}V^2\delta_{ij}\right), \tag{2.13}$$

$$\begin{aligned}
Y_{0|ijk\ell}(\mathbf{V}) &= V_iV_jV_kV_\ell - \frac{1}{d+4}V^2(V_iV_j\delta_{k\ell} + V_iV_k\delta_{j\ell} + V_iV_\ell\delta_{jk} + V_jV_k\delta_{i\ell} + V_jV_\ell\delta_{ik} + V_kV_\ell\delta_{ij}) \\
&\quad + \frac{1}{(d+2)(d+4)}V^4(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \\
&= V_iV_jV_kV_\ell - \frac{1}{d+4}[Y_{2|ij}(\mathbf{V})\delta_{k\ell} + Y_{2|ik}(\mathbf{V})\delta_{j\ell} + Y_{2|i\ell}(\mathbf{V})\delta_{jk} + Y_{2|jk}(\mathbf{V})\delta_{i\ell} \\
&\quad + Y_{2|j\ell}(\mathbf{V})\delta_{ik} + Y_{2|k\ell}(\mathbf{V})\delta_{ij}] - \frac{1}{d(d+2)}V^4(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}). \tag{2.14}
\end{aligned}$$

Here we will use the notation $M_{2r|i_1i_2...i_s} = \mathcal{M}[Y_{2r|i_1i_2...i_s}]$ and $J_{2r|i_1i_2...i_s} = \mathcal{J}[Y_{2r|i_1i_2...i_s}]$ for the associated moments and collisional moments, respectively. Note that $M_{0|0} = n$, $J_{0|0} = 0$ (conservation of mass), $M_{0|i} = 0$ (by definition of the peculiar velocity), $J_{0|i} = 0$ (conservation of momentum) and $M_{2|0} = pd/m$, where $p = nT$ is the hydrostatic pressure, T being the granular temperature. Moreover, $M_{0|ij} = (P_{ij} - p\delta_{ij})/m$, where P_{ij} is the pressure tensor and $M_{2|i} = 2q_i/m$, where $\mathbf{q}$ is the heat flux vector. The moment $M_{2|0}$, the number density n and the flow velocity $\mathbf{u}$ are the hydrodynamic fields, while the moments $M_{0|ij}$ and $M_{2|i}$ constitute the momentum and energy fluxes, respectively. The remaining third degree moments $M_{0|ijk}$ and the moments of a degree $k \geqslant 4$ are not directly related to the hydrodynamic description, but they are useful to provide information about the velocity distribution function. In particular, the moment $M_{4|0}$ is related to the fourth cumulant a_2 as

$$a_2 = \frac{m^2}{d(d+2)nT^2}M_{4|0} - 1, \tag{2.15}$$

while the moments $M_{0|ijk}$, $M_{0|ijk\ell}$ and $M_{2|ij}$ measure the degree of anisotropy of the velocity distribution.

As in the elastic case, the mathematical structure of the collision operator (2.2) implies that a collisional moment of degree k can be expressed in terms of velocity moments of a degree less than or equal to k. More specifically, the choice of the polynomials $Y_{2r|\bar{s}}(\mathbf{V})$, where we have introduced the short-hand notation $\bar{s} \equiv i_1i_2...i_s$, yields the following structural form for the collisional moments $J_{2r|\bar{s}}$:

$$J_{2r|\bar{s}} = -\nu_{2r|s}M_{2r|\bar{s}} + \sum_{r',r'',\bar{s}',\bar{s}''}^{\dagger}\lambda_{r'r''|\bar{s}'\bar{s}''\bar{s}}M_{2r'|\bar{s}'}M_{2r''|\bar{s}''}, \tag{2.16}$$

where the dagger in the summation denotes the constraints $2(r' + r'') + s' + s'' = 2r + s$, $2r' + s' \geqslant 2$, and $2r'' + s'' \geqslant 2$. Since the first term on the right-hand side of equation (2.16) is linear, $\nu_{2r|s}$ represents the eigenvalue of the linearized collision operator corresponding to the eigenfunction $Y_{2r|\bar{s}}(\mathbf{V})$.

Let us now display the explicit expressions for the collisional moments $J_{2r|i_1i_2...i_s}$ for $k = 2r + s \leqslant 4$. We start with the second degree moments.

2.1. Second degree collisional moments

The second degree collisional moments were already evaluated in [27]. They are given by

$$J_{2|0} = -\nu_{2|0}M_{2|0}, \qquad J_{0|ij} = -\nu_{0|2}M_{0|ij}, \tag{2.17}$$

where the expressions for the eigenvalues $\nu_{2|0}$ and $\nu_{0|2}$ are

$$\nu_{2|0} = \frac{d+2}{4d}(1-\alpha^2)\nu_0, \tag{2.18}$$

$$\nu_{0|2} = \frac{(1+\alpha)(d+1-\alpha)}{2d}\nu_0 = \nu_{2|0} + \frac{(1+\alpha)^2}{4}\nu_0. \tag{2.19}$$

The quantity $\nu_{2|0}$ is not but the cooling rate, i.e., the rate of change of the granular temperature due to the inelasticity of the collisions. The eigenvalue $\nu_{0|2}$ is the collision frequency associated with the NS shear viscosity and reduces to ν_0 in the elastic limit. The second equality in equation (2.19) decomposes $\nu_{0|2}$ into the part inherent to the collisional cooling plus the genuine part of the momentum collisional transfer. As shown below, a similar decomposition can be carried out for the eigenvalues $\nu_{2r|s}$.

2.2. Third degree collisional moments

The evaluation of the third degree collisional moments $J_{2|i}$ and $J_{0|ijk}$ is performed in the appendix. The results are

$$J_{2|i} = -\nu_{2|1} M_{2|i}, \qquad J_{0|ijk} = -\nu_{0|3} M_{0|ijk}, \tag{2.20}$$

where

$$\nu_{2|1} = \frac{(1+\alpha)\,[5d + 4 - \alpha(d+8)]}{8d}\nu_0 = \frac{3}{2}\nu_{2|0} + \frac{(1+\alpha)^2 (d-1)}{4d}\nu_0, \tag{2.21}$$

$$\nu_{0|3} = \frac{3}{2}\nu_{0|2}. \tag{2.22}$$

Equation (2.21) was first obtained in [27]. The eigenvalue $\nu_{2|1}$ is the collision frequency associated with the NS thermal conductivity. It is interesting to note that $\left(\nu_{2|1} - \frac{3}{2}\nu_{2|0}\right)\big/(\nu_{0|2} - \nu_{2|0}) = (d-1)/d$, which generalizes the simple relationship, holding for elastic Maxwell models, between the collision frequencies associated with the thermal conductivity and the shear viscosity. An even simpler extension is provided by equation (2.22).

2.3. Fourth degree collisional moments

The fourth degree collisional moments are also worked out in the appendix. They can be written as

$$J_{4|0} = -\nu_{4|0} M_{4|0} + \lambda_1 n^{-1} M_{2|0}^2 - \lambda_2 n^{-1} M_{0|ij} M_{0|ji}, \tag{2.23}$$

$$J_{2|ij} = -\nu_{2|2} M_{2|ij} + \lambda_3 n^{-1} M_{2|0} M_{0|ij} - \lambda_4 n^{-1}\left(M_{0|ik} M_{0|kj} - \frac{1}{d} M_{0|k\ell} M_{0|\ell k}\delta_{ij}\right), \tag{2.24}$$

$$\begin{aligned}
J_{0|ijk\ell} = {} & -\nu_{0|4} M_{0|ijk\ell} + \lambda_5 n^{-1}\bigg[M_{0|ij} M_{0|k\ell} + M_{0|ik} M_{0|j\ell} + M_{0|i\ell} M_{0|jk} \\
& - \frac{2}{d+4}(M_{0|ip} M_{0|pj}\delta_{k\ell} + M_{0|ip} M_{0|pk}\delta_{j\ell} + M_{0|ip} M_{0|p\ell}\delta_{jk} \\
& + M_{0|jp} M_{0|pk}\delta_{i\ell} + M_{0|jp} M_{0|p\ell}\delta_{ik} + M_{0|kp} M_{0|p\ell}\delta_{ij}) \\
& + \frac{2}{(d+2)(d+4)} M_{0|pq} M_{0|qp}(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\bigg].
\end{aligned} \tag{2.25}$$

In equations (2.23)–(2.25), the usual summation convention over repeated indices is assumed. The collision frequencies (or eigenvalues) $\nu_{2r|s}$ and the cross coefficients λ_i are given by

$$\begin{aligned}
\nu_{4|0} &= \frac{(1+\alpha)[12d + 9 - \alpha(4d+17) + 3\alpha^2 - 3\alpha^3]}{16d}\nu_0 \\
&= 2\nu_{2|0} + \frac{(1+\alpha)^2(4d - 7 + 6\alpha - 3\alpha^2)}{16d}\nu_0,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
\nu_{2|2} &= \frac{(1+\alpha)[7d^2 + 31d + 18 - \alpha(d^2 + 14d + 34) + 3\alpha^2(d+2) - 6\alpha^3]}{8d(d+4)}\nu_0 \\
&= 2\nu_{2|0} + \frac{(1+\alpha)^2[3d^2 + 7d - 14 + 3\alpha(d+4) - 6\alpha^2]}{8d(d+4)}\nu_0,
\end{aligned} \tag{2.27}$$

$$\nu_{0|4} = \frac{(1+\alpha)[2d^3 + 21d^2 + 61d + 39 - 3\alpha(d+3)(d+5) + 3\alpha^2(d+3) - 3\alpha^3]}{2d(d+4)(d+6)}\nu_0$$

$$= 2\nu_{2|0} + \frac{(1+\alpha)^2[d^3 + 9d^2 + 17d - 9 + 3\alpha(d+4) - 3\alpha^2]}{2d(d+4)(d+6)}\nu_0, \tag{2.28}$$

$$\lambda_1 = \frac{(1+\alpha)^2(d+2)(4d - 1 - 6\alpha + 3\alpha^2)}{16d^2}\nu_0, \tag{2.29}$$

$$\lambda_2 = \frac{(1+\alpha)^2(1 + 6\alpha - 3\alpha^2)}{8d}\nu_0, \tag{2.30}$$

$$\lambda_3 = \frac{(1+\alpha)^2[d^2 + 5d - 2 - 3\alpha(d+4) + 6\alpha^2]}{8d^2}\nu_0, \tag{2.31}$$

$$\lambda_4 = \frac{(1+\alpha)^2[2 - d + 3\alpha(d+4) - 6\alpha^2]}{4d(d+4)}\nu_0, \tag{2.32}$$

$$\lambda_5 = \frac{(1+\alpha)^2[d^2 + 7d + 9 - 3\alpha(d+4) + 3\alpha^2]}{2d(d+4)(d+6)}\nu_0. \tag{2.33}$$

Equations (2.26) and (2.29) coincide with the results of [27].

We have checked that, in the elastic case ($\alpha = 1$) and for three-dimensional systems ($d = 3$), all the expressions reported in this section reduce to known results [36, 37]. In the one-dimensional elastic case ($d = 1, \alpha = 1$), the gas behaves as an ideal gas because a collision is equivalent to exchanging the labels of both colliding particles. As a consequence, $J_{2r|\bar{s}} = 0$. It is easy to check that equations (2.17)–(2.33) are consistent with this property since the coefficients affecting the non-vanishing moments are zero, i.e., $\nu_{2|1} = \nu_{4|0} = \lambda_1 = 0$. Moreover, for one-dimensional inelastic gases ($d = 1, \alpha < 1$), our expressions for $\nu_{2|0}$, $\nu_{2|1}$, $\nu_{4|0}$ and λ_1 agree with the results derived by Ben-Naim and Krapivsky [12], who obtained the exact expressions for all the collisional moments, namely $J_{2r|0}$ and $J_{2r|x}$.

While the α dependence of the second and third degree eigenvalues, equations (2.18), (2.19), (2.21) and (2.22), is relatively simple, that of the fourth degree eigenvalues (2.26)–(2.28) and the cross coefficients (2.29)–(2.33) is more involved. Figure 1 shows the α dependence of the (reduced) eigenvalues $\nu_{4|0}^*$, $\nu_{2|2}^*$ and $\nu_{0|4}^*$, where $\nu_{2r|s}^* \equiv \nu_{2r|s}/\nu_0$, and the shifted eigenvalues $\omega_{4|0}$, $\omega_{2|2}$ and $\omega_{0|4}$, where we have called $\omega_{2r|s} \equiv \nu_{2r|s}^* - (r + s/2)\nu_{2|0}^*$, for $d = 2$ and $d = 3$. While $\nu_{0|4}^*$ decays monotonically as the inelasticity increases, the other two eigenvalues $\nu_{2|2}^*$ and $\nu_{4|0}^*$ start growing, reach a maximum, and then decay. The maximum value of $\nu_{2|2}^*$ occurs at $\alpha \simeq 0.40$ for $d = 2$ and at $\alpha \simeq 0.67$ for $d = 3$. In the case of $\nu_{4|0}^*$, the maximum occurs at $\alpha \simeq 0.18$ and $\alpha \simeq 0.30$ for $d = 2$ and $d = 3$, respectively. However, when the part associated with the cooling rate is subtracted from the bare eigenvalues, the resulting shifted quantities $\omega_{4|0}$, $\omega_{2|2}$ and $\omega_{0|4}$ exhibit a monotonic behavior. As shown in section 3, these shifted eigenvalues are the relevant ones in the time relaxation of the *scaled* moments in the free cooling problem. Therefore, the decrease of $\omega_{4|0}$, $\omega_{2|2}$ and $\omega_{0|4}$ implies that the characteristic relaxation times of the (scaled) fourth degree moments toward their asymptotic values increase with dissipation.

It is instructive to compare the fourth degree eigenvalues with the second and third degree eigenvalues. In the elastic case, one has $\omega_{4|0} = \omega_{2|1} < \omega_{0|2} = \omega_{2|2} < \omega_{0|3} < \omega_{0|4}$ for $d = 2$ and $\omega_{4|0} = \omega_{2|1} < \omega_{0|2} < \omega_{2|2} < \omega_{0|3} < \omega_{0|4}$ for $d = 3$. We have observed that inelasticity breaks the degeneracy $\omega_{4|0} = \omega_{2|1}$ for both dimensionalities (yielding $\omega_{4|0} < \omega_{2|1}$) and the degeneracy $\omega_{0|2} = \omega_{2|2}$ for $d = 2$ (yielding $\omega_{2|2} < \omega_{0|2}$). The inelasticity also affects the ordering of the eigenvalues: for $d = 2$ one has $\omega_{4|0} < \omega_{2|1} < \omega_{2|2} < \omega_{0|2} < \omega_{0|3} < \omega_{0|4}$
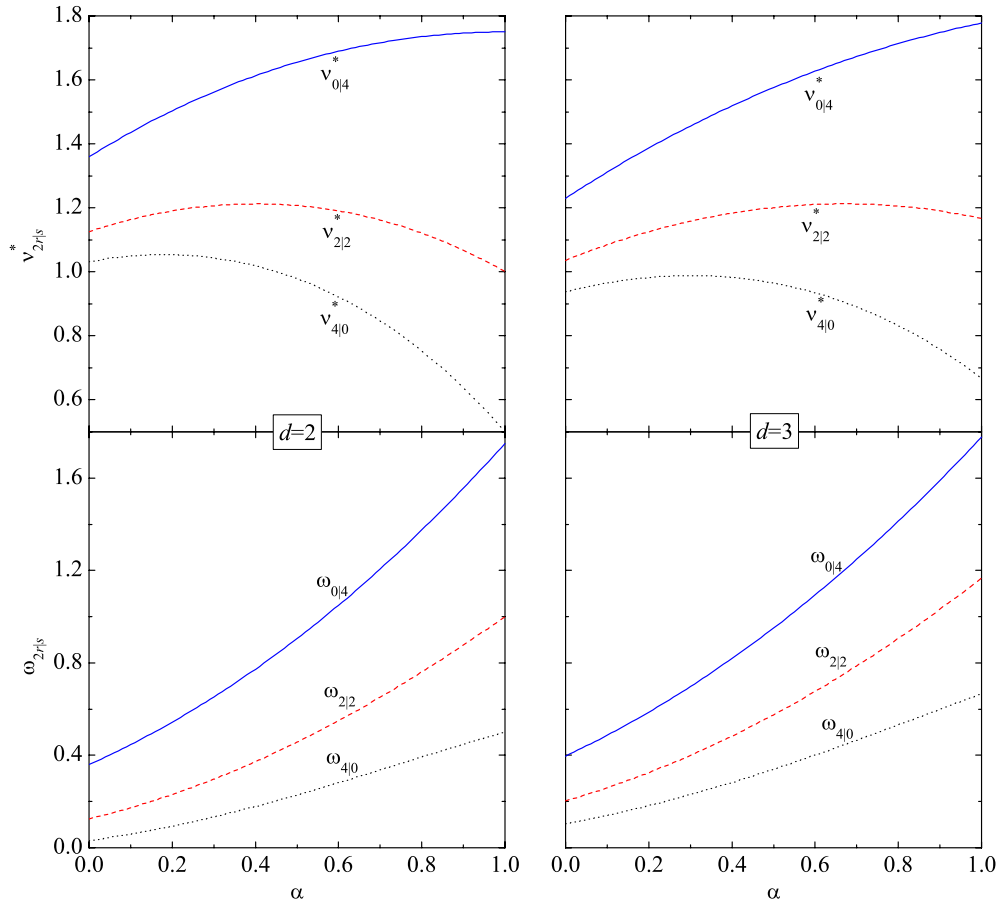
Figure 1. Plot of the (reduced) fourth degree eigenvalues $\nu^*_{0|4}$, $\nu^*_{2|2}$ and $\nu^*_{4|0}$ (top panels) and of the shifted eigenvalues $\omega_{0|4}$, $\omega_{2|2}$ and $\omega_{4|0}$ (bottom panels) as functions of the coefficient of restitution. The left and right panels correspond to $d = 2$ and $d = 3$, respectively.

if $0.17 < \alpha < 1$ and $\omega_{4|0} < \omega_{2|1} < \omega_{2|2} < \omega_{0|2} < \omega_{0|4} < \omega_{0|3}$ if $0 < \alpha < 0.17$; for $d = 3$ the ordering is $\omega_{4|0} < \omega_{2|1} < \omega_{0|2} < \omega_{2|2} < \omega_{0|3} < \omega_{0|4}$ if $0.43 < \alpha < 1$ and $\omega_{4|0} < \omega_{2|1} < \omega_{2|2} < \omega_{0|2} < \omega_{0|3} < \omega_{0|4}$ if $0 < \alpha < 0.43$. Since, except $\omega_{4|0}$, these quantities are related to moments which vanish in isotropic states, the fact that $\omega_{4|0}$ is the smallest one implies that (as expected on physical grounds) the characteristic time needed to achieve an isotropic state is shorter than the one needed to reach the asymptotic state.

Let us consider now the (reduced) cross coefficients $\lambda^*_i \equiv \lambda_i / \nu_0$ ($i = 1, \ldots, 5$), which measure the coupling of the second degree moments to the evolution of the fourth degree moments. Their dependence on dissipation is shown in figure 2. It is apparent that the effect of inelasticity on λ^*_i is more pronounced than on the fourth degree eigenvalues. For elastic collisions, $\lambda^*_3 = 0 < \lambda^*_5 < \lambda^*_4 = \lambda^*_2 = \lambda^*_1$ for $d = 2$ and $\lambda^*_5 < \lambda^*_3 < \lambda^*_4 = \lambda^*_2 < \lambda^*_1$ for $d = 3$. This ordering changes with inelasticity. Moreover, λ^*_1, λ^*_2 and λ^*_4 significantly decrease with increasing dissipation, λ^*_3 has a non-monotonic behavior, and λ^*_5 is nearly constant. Note that the coefficient λ_4 does not actually play any role in $d = 2$ since the combination $M_{0|ik}M_{0|kj} - \frac{1}{d}M_{0|k\ell}M_{0|\ell k}\delta_{ij}$ appearing in the collisional moment $J_{2|i}$, cf equation (2.24), vanishes in the two-dimensional case.
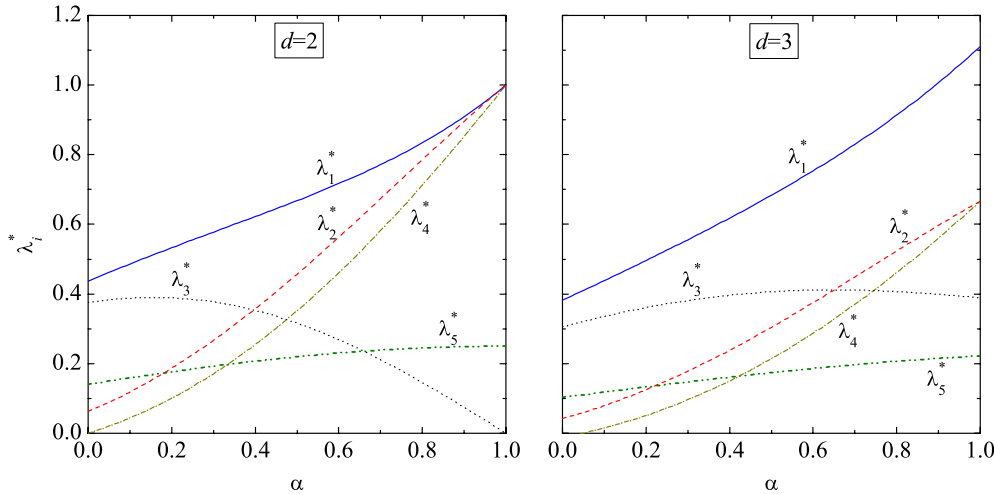
Figure 2. Plot of the (reduced) cross coefficients λ_i^*, as functions of the coefficient of restitution. The left and right panels correspond to $d = 2$ and $d = 3$, respectively.

3. Relaxation to the homogeneous cooling state

The results derived in the preceding section can be applied to several interesting situations. Here we will consider the most basic problem, namely the time evolution of the moments of degree less than or equal to 4 (both isotropic and anisotropic) in the *homogeneous* free cooling state [1]. In that case, the Boltzmann equation (2.1) becomes

$$\partial_t f(\mathbf{v}, t) = J[\mathbf{v}|f, f], \tag{3.1}$$

which must be complemented with a given initial condition $f(\mathbf{v}, 0)$. Since the collisions are inelastic, the granular temperature $T(t)$ monotonically decays in time and so a steady state does not exist. In the context of IMM, it has been proven [24, 25] that, provided that $f(\mathbf{v}, 0)$ has a finite moment of some degree higher than 2, $f(\mathbf{v}, t)$ asymptotically tends toward a self-similar solution of the form

$$f(\mathbf{v}, t) \rightarrow n[v_0(t)]^{-d} \phi(V/v_0(t)), \tag{3.2}$$

where $v_0(t) \equiv \sqrt{2T(t)/m}$ is the thermal speed and $\phi(c)$ is an isotropic distribution that is only known in the one-dimensional case [17]. According to equation (3.2), the *scaled* moments

$$M_{2r|\bar{s}}^*(t) \equiv n^{-1}[v_0(t)]^{-(2r+s)} M_{2r|\bar{s}}(t) \tag{3.3}$$

must tend asymptotically to

$$M_{2r|\bar{s}}^*(t) \rightarrow \mu_{2r|\bar{s}} \equiv \int d\mathbf{c}\, Y_{2r|\bar{s}}(\mathbf{c}) \phi(c). \tag{3.4}$$

Due to the isotropy of $\phi(c)$, then $\mu_{2r|\bar{s}} = 0$ unless $s = 0$. Moreover, it is known that the scaled distribution $\phi(c)$ exhibits an algebraic high velocity tail [14, 15, 18, 19] of the form $\phi(c) \sim c^{-d-\gamma(\alpha)}$, so that the moments $\mu_{2r|0}$ diverge if $2r \geqslant \gamma(\alpha)$. The quantity $\gamma(\alpha)$ obeys a transcendental equation whose solution is always $\gamma(\alpha) > 4$, except for $d = 1$. Consequently, for any value of α and $d \geqslant 2$, the scaled moment $M_{4|0}^*(t)$ goes to a well-defined value $\mu_{4|0}$, while the remaining scaled moments of degree equal to or less than 4 are anisotropic (except of course $M_{0|0}^* = 1$ and $M_{2|0}^* = d/2$) and so they tend to zero. The main goal of this section

is to analyze in detail the relaxation of the second, third and fourth degree moments (both isotropic and anisotropic) toward their asymptotic values.

Taking velocity moments in both sides of equation (3.1) one has

$$\partial_t M_{2r|\bar{s}} = J_{2r|\bar{s}}. \tag{3.5}$$

In particular,

$$\partial_t M_{2|0} = -\nu_{2|0} M_{2|0}. \tag{3.6}$$

Since $M_{2|0} = dnT/m$, equation (3.6) is the equation for the time evolution of the granular temperature and $\nu_{2|0}$ is the cooling rate. The solution of equation (3.6) is

$$T(t) = \frac{T(0)}{[1 + \nu_{2|0}(0)t/2]^2}, \tag{3.7}$$

where $T(0)$ is the initial temperature and $\nu_{2|0}(0) \propto T^{1/2}(0)$ is the initial cooling rate. Equation (3.7) is not but Haff's law [1].

Let us consider now the scaled moments (3.3). In that case, from equations (3.5) and (3.6) one simply obtains

$$\partial_\tau M_{2r|\bar{s}}^* = J_{2r|\bar{s}}^* + \frac{2r+s}{2} \nu_{2|0}^* M_{2r|\bar{s}}^*, \tag{3.8}$$

where $J_{2r|\bar{s}}^* \equiv J_{2r|\bar{s}}/\nu_0 n v_0^{2r+s}$ and

$$\tau = \int_0^t dt' \, \nu_0(t') \tag{3.9}$$

measures time as the number of (effective) collisions per particle. The effect of the second term on the right-hand side of equation (3.8) is to shift the eigenvalues $\nu_{2r|s}^*$ to $\omega_{2r|s} = \nu_{2r|s}^* - (r + s/2)\nu_{2|0}^*$. For instance,

$$\partial_\tau M_{4|0}^* = -\omega_{4|0} M_{4|0}^* + \lambda_1^* \frac{d^2}{4} - \lambda_2^* M_{0|ij}^* M_{0|ji}^*. \tag{3.10}$$

3.1. The one-dimensional case

In the one-dimensional case, the only scaled moments of degree equal to or less than 4 are (apart from $M_{0|0}^* = 1$ and $M_{2|0}^* = \frac{1}{2}$) $M_{2|x}^*$ and $M_{4|0}^*$. Their evolution equations are

$$\partial_\tau M_{2|x}^* = 0, \tag{3.11}$$

$$\partial_\tau M_{4|0}^* = \tfrac{3}{16}(1-\alpha^2)^2 \left(M_{4|0}^* + \tfrac{3}{4}\right). \tag{3.12}$$

The solution of equation (3.12) is

$$M_{4|0}^*(\tau) = M_{4|0}^*(0) \, e^{\frac{3}{16}(1-\alpha^2)^2 \tau} + \tfrac{3}{4}\left[e^{\frac{3}{16}(1-\alpha^2)^2 \tau} - 1\right]. \tag{3.13}$$

This solution shows that the (scaled) fourth degree moment monotonically increases with time, i.e., $\mu_{4|0} = \infty$. This is consistent with the exact HCS solution found by Baldassarri et al [17], namely

$$\phi(c) = \frac{2^{3/2}}{\pi} \frac{1}{(1 + 2c^2)^2}. \tag{3.14}$$

On the other hand, equation (3.11) shows that $M_{2|x}^*(\tau) = M_{2|x}^*(0)$, i.e., if the initial state is anisotropic with $M_{2|x}(0) \neq 0$ then one has $M_{2|x}(t) = M_{2|x}(0)[T(t)/T(0)]^{3/2}$. The constancy of $M_{2|x}^*$ implies that any initial anisotropy does not vanish in the scaled velocity distribution function for long times. As a consequence, while the distribution (3.14) represents

the asymptotic form $\phi(c)$ for a wide class of *isotropic* initial conditions, it cannot be reached, strictly speaking, for any *anisotropic* initial state. Whether or not there exists a generalization of (3.14) for anisotropic states is, to the best of our knowledge, an open problem. Since the symmetry of the distribution (3.14) implies that $M_{2|x}^* \equiv \langle c_x^3 \rangle = 0$ but the average $\langle |c|^3 \rangle$ diverges, a small correction to the form (3.14) could accommodate a finite value of $\langle c_x^3 \rangle$.

3.2. The two-dimensional case

As is known, one-dimensional systems are generally not very realistic and so they can exhibit peculiar properties. However, two-dimensional systems are usually considered as representative of the features present in real systems.

For the sake of simplicity, in the remainder of this section we will consider the two-dimensional case. The set of independent moments of second, third and fourth degree will be taken as

$$\{M_{0|xx}^*, M_{0|xy}^*\}, \tag{3.15}$$

$$\{M_{2|x}^*, M_{2|y}^*, M_{0|xxx}^*, M_{0|xxy}^*\}, \tag{3.16}$$

$$\{M_{4|0}^*, M_{2|xx}^*, M_{2|xy}^*, M_{0|xxxx}^*, M_{0|xxxy}^*\}, \tag{3.17}$$

respectively. The remaining moments are simply related to the above ones as $M_{0|yy}^* = -M_{0|xx}^*$, $M_{0|xyy}^* = -M_{0|xxx}^*$, $M_{0|yyy}^* = -M_{0|xxy}^*$, $M_{2|yy}^* = -M_{2|xx}^*$, $M_{0|yyyy}^* = -M_{0|xxyy}^* = M_{0|xxxx}^*$, $M_{0|xyyy}^* = -M_{0|xxxy}^*$. From equation (3.8), it is easy to obtain the time dependence of the (scaled) second and third degree moments:

$$M_{0|ij}^*(\tau) = M_{0|ij}^*(0)\, e^{-\omega_{0|2}\tau}, \tag{3.18}$$

$$M_{2|i}^*(\tau) = M_{2|i}^*(0)\, e^{-\omega_{2|1}\tau}, \quad M_{0|ijk}^*(\tau) = M_{0|ijk}^*(0)\, e^{-\omega_{0|3}\tau}. \tag{3.19}$$

In the case of the fourth degree moments, one has to deal with inhomogeneous linear differential equations involving the second degree moments. The solutions are

$$
\begin{aligned}
M_{4|0}^*(\tau) = {}& M_{4|0}^*(0)\, e^{-\omega_{4|0}\tau} + \frac{\lambda_1^*}{\omega_{4|0}}(1 - e^{-\omega_{4|0}\tau}) \\
& - \frac{2\lambda_2^*}{2\omega_{0|2} - \omega_{4|0}} \left[M_{0|xx}^{*2}(0) + M_{0|xy}^{*2}(0) \right] (e^{-\omega_{4|0}\tau} - e^{-2\omega_{0|2}\tau}),
\end{aligned} \tag{3.20}
$$

$$M_{2|ij}^*(\tau) = M_{2|ij}^*(0)\, e^{-\omega_{2|2}\tau} + \frac{\lambda_3^*}{\omega_{0|2} - \omega_{2|2}} M_{0|ij}^*(0)(e^{-\omega_{2|2}\tau} - e^{-\omega_{0|2}\tau}), \tag{3.21}$$

$$M_{0|xxxx}^*(\tau) = M_{0|xxxx}^*(0)\, e^{-\omega_{0|4}\tau} + \frac{3}{2}\frac{\lambda_5^*}{2\omega_{0|2} - \omega_{0|4}} \left[M_{0|xx}^{*2}(0) - M_{0|xy}^{*2}(0) \right] (e^{-\omega_{0|4}\tau} - e^{-2\omega_{0|2}\tau}), \tag{3.22}$$

$$M_{0|xxxy}^*(\tau) = M_{0|xxxy}^*(0)\, e^{-\omega_{0|4}\tau} + \frac{3\lambda_5^*}{2\omega_{0|2} - \omega_{0|4}} M_{0|xx}^*(0) M_{0|xy}^*(0)(e^{-\omega_{0|4}\tau} - e^{-2\omega_{0|2}\tau}). \tag{3.23}$$

Equations (3.18)–(3.23) show that all the moments, except $M_{4|0}^*$, tend to zero for sufficiently long times. The asymptotic expression of $M_{4|0}^*$ is

$$M_{4|0}^* \to \mu_{4|0} = \frac{\lambda_1^*}{\omega_{4|0}} = 2\frac{7 - 6\alpha + 3\alpha^2}{1 + 6\alpha - 3\alpha^2}, \tag{3.24}$$

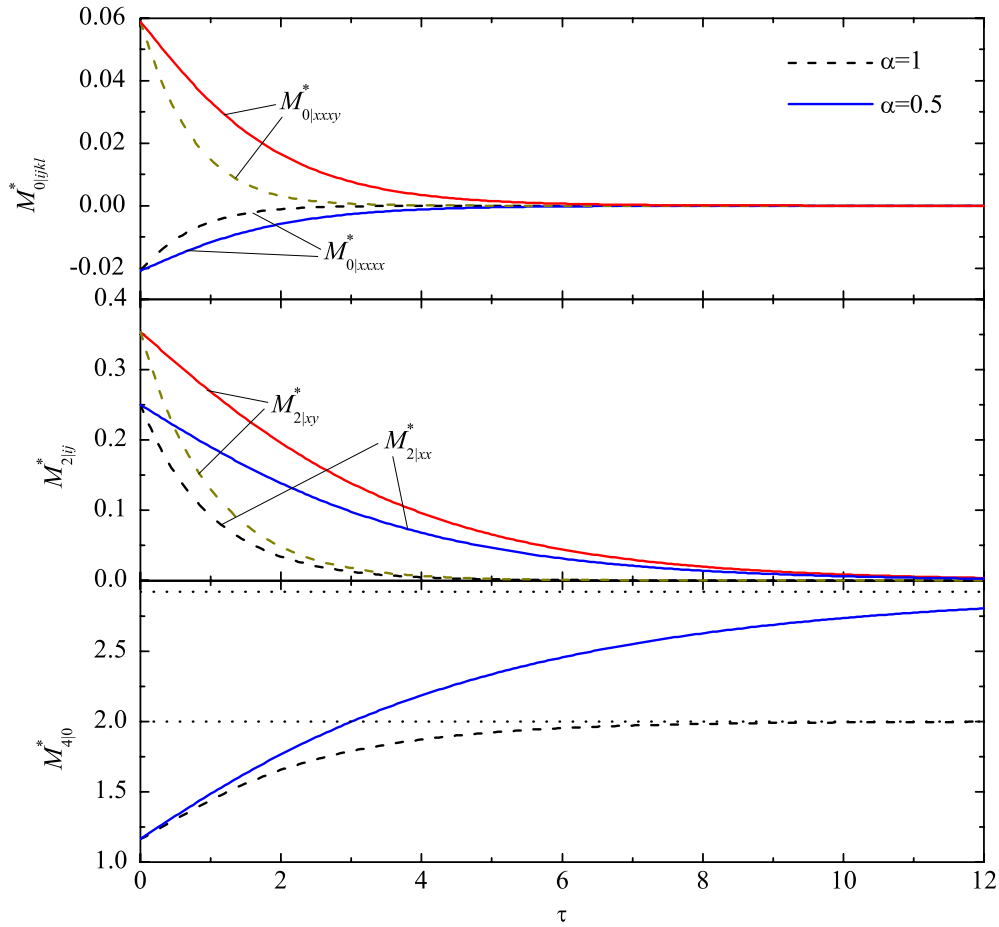which agrees with previous results [27].

Figure 3. Time evolution of the scaled fourth degree moments in the free cooling of a two-dimensional IMM with $\alpha = 0.5$ (solid lines). The time evolution to equilibrium of an elastic system ($\alpha = 1$) is also shown (dashed lines). In both cases the initial state is described by equation (3.25). The horizontal dotted lines in the bottom panel indicate the corresponding asymptotic HCS values $\mu_{4|0}$.

As an illustration, let us analyze the time evolution of the scaled fourth degree moments (3.17) for the following initial anisotropic distribution:

$$f(\mathbf{v}, 0) = \frac{n}{3}[\delta(\mathbf{V} - \mathbf{V}_1) + \delta(\mathbf{V} - \mathbf{V}_2) + \delta(\mathbf{V} - \mathbf{V}_3)], \tag{3.25}$$

where $\mathbf{V}_1 = v_0(0)\hat{\mathbf{x}}$, $\mathbf{V}_2 = (v_0(0)/\sqrt{2})\hat{\mathbf{y}}$ and $\mathbf{V}_3 = -\mathbf{V}_1 - \mathbf{V}_2$. Here, $v_0(0) = \sqrt{2T(0)/m}$ is the initial thermal speed, where the initial temperature $T(0)$ is arbitrary. Figure 3 shows the evolution of the moments (3.17) for two values of the coefficient of restitution: $\alpha = 1$ (elastic system) and $\alpha = 0.5$ (strongly inelastic system). It is quite apparent that the number of collisions needed to reach the HCS values increases with the inelasticity, as expected from figure 1. In the particular case of $\alpha = 0.5$, the relaxation times are about twice the ones corresponding to $\alpha = 1$. Moreover, since $\omega_{0|4} > \omega_{2|2} > \omega_{4|0}$, we observe that the moments $M^*_{0|ijk\ell}$ tend to zero more rapidly than the moments $M^*_{2|ij}$, and that the isotropic moment $M^*_{4|0}$ reaches its asymptotic value more slowly than the anisotropic moments.

4. Discussion

As Maxwell already realized [38], scattering models where the collision rate of two particles approaching each other with a relative velocity $\mathbf{g}$ is independent of the magnitude of $\mathbf{g}$ allows one to evaluate exactly the collisional moments without the explicit knowledge of the velocity distribution function. In the conventional case of ordinary gases of particles colliding elastically, Maxwell models are useful to find non-trivial exact solutions to the Boltzmann equation in far from equilibrium situations [37]. Needless to say, the introduction of inelasticity through a constant coefficient of normal restitution $\alpha \leqslant 1$ opens up new perspectives for exact results, including the elastic case as a special limit ($\alpha = 1$). This justifies the growing interest in IMM by physicists and mathematicians alike in the past few years.

The choice of the Ikenberry polynomials [36] $Y_{2r|\bar{s}}$ of degree $2r + s$ allows one to express the corresponding collisional moments $J_{2r|\bar{s}}$ in the form (2.16): an eigenvalue $-\nu_{2r|s}$ times the velocity moment $M_{2r|\bar{s}}$ plus a bilinear combination of moments of degree less than $2r + s$. In particular, $\nu_{2|0}$ is the cooling rate of the gas. In this paper we have evaluated all the third and fourth degree collisional moments of the IMM defined by equation (2.2). In that context, the results are exact for arbitrary values of α and apply to any dimensionality d. Known results are recovered for three-dimensional elastic systems [36, 37] and for one-dimensional inelastic systems [12]. We have observed that some of the eigenvalues $\nu_{2r|s}$ do not have a monotonic dependence on α, while the shifted eigenvalues $\nu_{2r|s} - (r + s/2)\nu_{2|0}$ monotonically decrease with increasing inelasticity. Moreover, given a value of α and a degree $2r + s$, the eigenvalues $\nu_{2r|s}$ increase with s. This means that the larger the anisotropy of a moment $M_{2r|\bar{s}}$ the higher its collisional rate of change. Although the above observations are based on the moments of degree $2r + s \leqslant 4$, we expect that they extend to moments of higher degree.

As a simple application of the results derived in section 2, we have studied the time evolution of the moments of degree equal to or less than 4 in the free cooling state, in which case the evolution of the moments scaled with the thermal speed is governed by the shifted eigenvalues. An interesting feature of the one-dimensional case is that the heat flux $q_x = (m/2)M_{2|x}$, when scaled with the thermal speed, does not change in time, so that an initial anisotropic distribution cannot evolve toward an asymptotic isotropic distribution. Thus, the exact solution found by Baldassarri *et al* [17] does not play a universal role, at least in a strict sense, unless the initial distribution is isotropic. On the other hand, we have found that all the anisotropic moments of degree equal to or less than 4 vanish in the long time limit for $d \geqslant 2$. However, this does not preclude the possibility that anisotropic moments of higher degree diverge for α sufficiently small. We plan to explore this possibility in the near future.

The explicit results provided in this paper can be useful for studying different problems. An important application is the exact derivation of the Burnett order constitutive equations for IMM, with explicit expressions of the associated transport coefficients as functions of d and α. This is possible because the determination of the Burnett order pressure tensor and heat flux requires the previous knowledge of the third and fourth degree collisional moments to Navier–Stokes order. Another interesting problem is the so-called simple or uniform shear flow, which is an intrinsically non-Newtonian state [39]. Apart from the rheological quantities, the results derived here allows one to analyze the time evolution of the fourth degree velocity moments toward their steady state values [40] and investigate their possible divergence, in a similar way to the analysis carried out in the elastic case [41]. Finally, the generalized transport coefficients characterizing small perturbations around the simple shear flow have been determined [42] and compared with those previously obtained for IHS [43] from a model kinetic equation.

Acknowledgments

Appendix A. Explicit evaluation of the collisional moments

In this appendix we give the details of the derivation, by using the property (2.6), of the collisional moments $J_{2r|\bar{s}} = \mathcal{J}[Y_{2r|\bar{s}}]$ associated with the Ikenberry polynomials of third and fourth degree. To carry out the calculations we will need the angular integrals

$$\int d\widehat{\boldsymbol{\sigma}}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^{2r+1}\widehat{\sigma}_i = B_{r+1}g^{2r}g_i, \tag{A.1}$$

$$\int d\widehat{\boldsymbol{\sigma}}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^{2r}\widehat{\sigma}_i\widehat{\sigma}_j = \frac{B_r}{2r+d}g^{2(r-1)}(2rg_ig_j + g^2\delta_{ij}), \tag{A.2}$$

$$\int d\widehat{\boldsymbol{\sigma}}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^{2r+1}\widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k = \frac{B_{r+1}}{2(r+1)+d}g^{2(r-1)}[2rg_ig_jg_k + g^2(\delta_{ij}g_k + \delta_{ik}g_j + \delta_{jk}g_i)], \tag{A.3}$$

$$\begin{aligned}\int d\widehat{\boldsymbol{\sigma}}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^{2r}\widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k\widehat{\sigma}_\ell &= \frac{B_r}{(2r+d)[d+2(r+1)]}\Big[4r(r-1)g^{2(r-2)}g_ig_jg_kg_\ell \\ &\quad + g^{2(r-1)}2r(g_ig_j\delta_{k\ell} + g_ig_k\delta_{j\ell} + g_ig_\ell\delta_{jk} + g_kg_j\delta_{i\ell} + g_\ell g_j\delta_{ik} + g_kg_\ell\delta_{ij}) \\ &\quad + g^{2r}(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell})\Big].\end{aligned} \tag{A.4}$$

Here, the coefficients B_r are [3]

$$B_r = \int d\widehat{\boldsymbol{\sigma}}(\widehat{\boldsymbol{\sigma}} \cdot \widehat{\mathbf{g}})^{2r} = \Omega_d\pi^{-1/2}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(r + \frac{1}{2}\right)}{\Gamma\left(r + \frac{d}{2}\right)}. \tag{A.5}$$

A.1. Third degree moments

We start by noting that the collision rule (2.7) implies that

$$\begin{aligned}V_{1i}''V_{1j}''V_{1k}'' &= V_{1i}V_{1j}V_{1k} - \frac{1+\alpha}{2}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})(V_{1i}V_{1j}\widehat{\sigma}_k + V_{1i}V_{1k}\widehat{\sigma}_j + V_{1j}V_{1k}\widehat{\sigma}_i) \\ &\quad + \left(\frac{1+\alpha}{2}\right)^2(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^2(V_{1i}\widehat{\sigma}_j\widehat{\sigma}_k + V_{1j}\widehat{\sigma}_i\widehat{\sigma}_k + V_{1k}\widehat{\sigma}_i\widehat{\sigma}_j) - \left(\frac{1+\alpha}{2}\right)^3(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3\widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k.\end{aligned} \tag{A.6}$$

Next, making use of equations (A.1)–(A.3), one obtains

$$\begin{aligned}\mathcal{J}[V_{1i}V_{1j}V_{1k}] &= -\frac{nv_0}{2d}\frac{1+\alpha}{2}\Big\{(d+2)\langle g_iV_{1j}V_{1k} + g_jV_{1i}V_{1k} + g_kV_{1i}V_{1j}\rangle \\ &\quad - \frac{1+\alpha}{2}[2\langle g_ig_jV_{1k} + g_ig_kV_{1j} + g_jg_kV_{1i}\rangle + \langle g^2(V_{1i}\delta_{jk} + V_{1j}\delta_{ik} + V_{1k}\delta_{ij})\rangle]\Big\},\end{aligned} \tag{A.7}$$

where the brackets are defined as

$$\langle h(\mathbf{V}_1, \mathbf{V}_2)\rangle \equiv \frac{1}{n^2}\int d\mathbf{V}_1\int d\mathbf{V}_2\, h(\mathbf{V}_1, \mathbf{V}_2)f(\mathbf{V}_1)f(\mathbf{V}_2) \tag{A.8}$$

and we have taken into account that $\langle g_i g_j g_k \rangle = \langle g^2 g_i \rangle = 0$. It is easy to obtain

$$n\langle g_i V_{1j} V_{1k} \rangle = n\langle g_i g_j V_{1k} \rangle = M_{0|ijk} + \frac{1}{d+2}(M_{2|i}\delta_{jk} + M_{2|j}\delta_{ik} + M_{2|k}\delta_{ij}), \tag{A.9}$$

$$n\langle g^2 V_{1i} \rangle = M_{2|i}. \tag{A.10}$$

Therefore,

$$\mathcal{J}[V_{1i} V_{1j} V_{1k}] = -\frac{\nu_0}{2d}\frac{1+\alpha}{2}\left[3(d+1-\alpha)M_{0|ijk}\right.$$
$$\left. + \frac{5d+4-\alpha(d+8)}{2(d+2)}(M_{2|i}\delta_{jk} + M_{2|j}\delta_{ik} + M_{2|k}\delta_{ij})\right]. \tag{A.11}$$

If one makes $j = k$ and sum over j one gets the first equality of equation (2.20) with $\nu_{2|1}$ given by equation (2.21). Also, by subtracting $(J_{2|i}\delta_{jk} + J_{2|j}\delta_{ik} + J_{2|k}\delta_{ij})/(d+2)$ from both sides of equation (A.11) one gets the second equality of equation (2.20) with $\nu_{0|3}$ given by equation (2.22).

A.2. Fourth degree moments

Now the starting point is the collision rule

$$V_{1i}'' V_{1j}'' V_{1k}'' V_{1\ell}'' = V_{1i} V_{1j} V_{1k} V_{1\ell} - \frac{1+\alpha}{2}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})(V_{1i} V_{1j} V_{1k}\widehat{\sigma}_\ell + V_{1i} V_{1j} V_{1\ell}\widehat{\sigma}_k + V_{1i} V_{1k} V_{1\ell}\widehat{\sigma}_j$$
$$+ V_{1j} V_{1k} V_{1\ell}\widehat{\sigma}_i) + \left(\frac{1+\alpha}{2}\right)^2 (\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^2 (V_{1i} V_{1j}\widehat{\sigma}_k\widehat{\sigma}_\ell + V_{1i} V_{1k}\widehat{\sigma}_j\widehat{\sigma}_\ell + V_{1i} V_{1\ell}\widehat{\sigma}_j\widehat{\sigma}_k$$
$$+ V_{1j} V_{1k}\widehat{\sigma}_i\widehat{\sigma}_\ell + V_{1j} V_{1\ell}\widehat{\sigma}_i\widehat{\sigma}_k + V_{1k} V_{1\ell}\widehat{\sigma}_i\widehat{\sigma}_j)$$
$$- \left(\frac{1+\alpha}{2}\right)^3 (\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3 (V_{1i}\widehat{\sigma}_j\widehat{\sigma}_k\widehat{\sigma}_\ell + V_{1j}\widehat{\sigma}_i\widehat{\sigma}_k\widehat{\sigma}_\ell + V_{1k}\widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_\ell + V_{1\ell}\widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k)$$
$$+ \left(\frac{1+\alpha}{2}\right)^4 (\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^4\widehat{\sigma}_i\widehat{\sigma}_j\widehat{\sigma}_k\widehat{\sigma}_\ell. \tag{A.12}$$

After integrating over $\widehat{\boldsymbol{\sigma}}$,

$$\mathcal{J}[V_{1i} V_{1j} V_{1k} V_{i\ell}] = -\frac{n\nu_0}{2d(d+4)(d+6)}\frac{1+\alpha}{2}\left\{(d+2)(d+4)(d+6)\langle g_i V_{1j} V_{1k} V_{1\ell}+\overset{(3)}{\cdots}\rangle\right.$$
$$- \frac{1+\alpha}{2}(d+4)(d+6)\left[2\langle g_i g_j V_{1k} V_{1\ell}+\overset{(5)}{\cdots}\rangle + \langle g^2(V_{1i} V_{ij}\delta_{k\ell}+\overset{(5)}{\cdots})\rangle\right]$$
$$+ 3(d+6)\left(\frac{1+\alpha}{2}\right)^2 \left[2\langle g_i g_j g_k V_{1\ell}+\overset{(3)}{\cdots}\rangle + \langle g^2[(g_i V_{1j} + g_j V_{1i})\delta_{k\ell}+\overset{(5)}{\cdots}]\rangle\right]$$
$$\left. - 3\left(\frac{1+\alpha}{2}\right)^3 \left[8\langle g_i g_j g_k g_\ell \rangle + 4\langle g^2(g_i g_j\delta_{k\ell}+\overset{(5)}{\cdots})\rangle + \langle g^4\rangle(\delta_{ij}\delta_{k\ell}+\overset{(2)}{\cdots})\right]\right\}, \tag{A.13}$$

where $\overset{(s)}{\cdots}$ denotes the s terms obtained from the canonical one by permutation of indices. Making $k = \ell$ and summing over k we obtain

$$\mathcal{J}[V_1^2 V_{1i} V_{1j}] = -\frac{n\nu_0}{2d(d+4)}\frac{1+\alpha}{2}\Bigg\{(d+2)(d+4)\langle 2(\mathbf{g}\cdot\mathbf{V}_1)V_{1i}V_{1j} + V^2(g_i V_{1j} + g_j V_{1i})\rangle$$

$$-\frac{1+\alpha}{2}(d+4)\big[2\langle V_1^2 g_i g_j + 2(\mathbf{g}\cdot\mathbf{V}_1)(g_i V_{1j} + g_j V_{1i})\rangle$$

$$+\langle(d+6)g^2 V_{1i}V_{1j} + g^2 V_1^2\delta_{ij}\rangle\big]$$

$$+3\left(\frac{1+\alpha}{2}\right)^2\big[(d+6)\langle g^2(g_i V_{1j} + g_j V_{1i})\rangle + 2\langle(\mathbf{g}\cdot\mathbf{V}_1)(2g_i g_j + g^2\delta_{ij})\rangle\big]$$

$$-3\left(\frac{1+\alpha}{2}\right)^3\langle 4g^2 g_i g_j + g^4\delta_{ij}\rangle\Bigg\}. \tag{A.14}$$

Next, by taking $i = j$ and summing over i, equation (A.14) yields

$$\mathcal{J}[V_1^4] = -\frac{n\nu_0}{2d}\frac{1+\alpha}{2}\Bigg\{4(d+2)\langle(\mathbf{g}\cdot\mathbf{V}_1)V_1^2\rangle - 2\frac{1+\alpha}{2}\big[(d+4)\langle g^2 V_1^2\rangle + 4\langle(\mathbf{g}\cdot\mathbf{V}_1)^2\rangle\big]$$

$$+12\left(\frac{1+\alpha}{2}\right)^2\langle(\mathbf{g}\cdot\mathbf{V}_1)g^2\rangle - 3\left(\frac{1+\alpha}{2}\right)^3\langle g^4\rangle\Bigg\}. \tag{A.15}$$

Now we express the averages in terms of the Ikenberry moments. Let us consider first the four-index averages:

$$n\langle g_i V_{1j} V_{1k} V_{1\ell}\rangle = M_{0|ijk\ell} + \frac{1}{d+4}(M_{2|ij}\delta_{k\ell} + \overset{(5)}{\cdots}) + \frac{1}{d(d+2)}M_{4|0}(\delta_{ij}\delta_{k\ell} + \overset{(2)}{\cdots}), \tag{A.16}$$

$$n\langle g_i g_j V_{1k} V_{1\ell}\rangle = M_{0|ijk\ell} + \frac{1}{d+4}(M_{2|ij}\delta_{k\ell} + \overset{(5)}{\cdots}) + \frac{1}{d(d+2)}M_{4|0}(\delta_{ij}\delta_{k\ell} + \overset{(2)}{\cdots})$$

$$+ n^{-1}M_{0|ij}M_{0|k\ell} + \frac{n^{-1}}{d}M_{2|0}(M_{0|ij}\delta_{k\ell} + M_{0|k\ell}\delta_{ij}) + \frac{n^{-1}}{d^2}M_{2|0}^2\delta_{ij}\delta_{k\ell}, \tag{A.17}$$

$$n\langle g_i g_j g_k V_{1\ell}\rangle = M_{0|ijk\ell} + \frac{1}{d+4}(M_{2|ij}\delta_{k\ell} + \overset{(5)}{\cdots}) + \frac{1}{d(d+2)}M_{4|0}(\delta_{ij}\delta_{k\ell} + \overset{(2)}{\cdots})$$

$$+ n^{-1}(M_{0|ij}M_{0|k\ell} + \overset{(2)}{\cdots}) + \frac{n^{-1}}{d}M_{2|0}(M_{0|ij}\delta_{k\ell} + \overset{(5)}{\cdots}) + \frac{n^{-1}}{d^2}M_{2|0}^2(\delta_{ij}\delta_{k\ell} + \overset{(2)}{\cdots}), \tag{A.18}$$

$$\langle g_i g_j g_k g_\ell\rangle = 2\langle g_i g_j g_k V_{1\ell}\rangle. \tag{A.19}$$

Summing over two repeated indices we get the two-index averages:

$$n\langle g^2 V_{1i} V_{1j}\rangle = n\langle V_1^2 g_i g_j\rangle = M_{2|ij} + \frac{1}{d}M_{4|0}\delta_{ij} + n^{-1}M_{2|0}\left(M_{0|ij} + \frac{1}{d}M_{2|0}\delta_{ij}\right), \tag{A.20}$$

$$n\langle g^2 g_i V_{1j}\rangle = n\langle(\mathbf{g}\cdot\mathbf{V}_1)g_i g_j\rangle = M_{2|ij} + \frac{1}{d}M_{4|0}\delta_{ij} + 2n^{-1}M_{0|ik}M_{0|kj}$$

$$+\frac{n^{-1}}{d}M_{2|0}\left[(d+4)M_{0|ij} + \frac{d+2}{d}M_{2|0}\delta_{ij}\right], \tag{A.21}$$

$$\langle g^2 g_i g_j\rangle = 2\langle g^2 g_i V_{1j}\rangle, \tag{A.22}$$

$$n\langle V_1^2 g_i V_{1j}\rangle = n\langle (\mathbf{g}\cdot\mathbf{V}_1)V_{1i}V_{1j}\rangle = M_{2|ij} + \frac{1}{d}M_{4|0}\delta_{ij}, \tag{A.23}$$

$$n\langle (\mathbf{g}\cdot\mathbf{V}_1)g_i V_{1j}\rangle = M_{2|ij} + \frac{1}{d}M_{4|0}\delta_{ij} + n^{-1}M_{0|ik}M_{0|kj} + \frac{n^{-1}}{d}M_{2|0}\left[2M_{0|ij} + \frac{1}{d}M_{2|0}\delta_{ij}\right]. \tag{A.24}$$

Summing again,

$$n\langle g^2 V_1^2\rangle = M_{4|0} + n^{-1}M_{2|0}^2, \tag{A.25}$$

$$n\langle (\mathbf{g}\cdot\mathbf{V}_1)V_1^2\rangle = M_{4|0}, \tag{A.26}$$

$$n\langle (\mathbf{g}\cdot\mathbf{V}_1)^2\rangle = M_{4|0} + n^{-1}M_{0|ij}M_{0|ji} + \frac{n^{-1}}{d}M_{2|0}^2, \tag{A.27}$$

$$n\langle (\mathbf{g}\cdot\mathbf{V}_1)g^2\rangle = M_{4|0} + 2n^{-1}M_{0|ij}M_{0|ji} + n^{-1}\frac{d+2}{d}M_{2|0}^2, \tag{A.28}$$

$$\langle g^4\rangle = 2\langle (\mathbf{g}\cdot\mathbf{V}_1)g^2\rangle. \tag{A.29}$$

Substituting equations (A.25)–(A.29) into equation (A.15) one obtains equation (2.23). Analogously, from equations (A.14) and (A.13) one obtains, after some algebra, equations (2.24) and (2.25), respectively.

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